F1 Jan 2014 (IAL) (MA)

Q1(a) $f(3) = 1.2256...$
$f(4) = -4.125$

} change in sign between $x=3$ and $x=4$ ∴ a root lies between $x=3$ and $x=4$

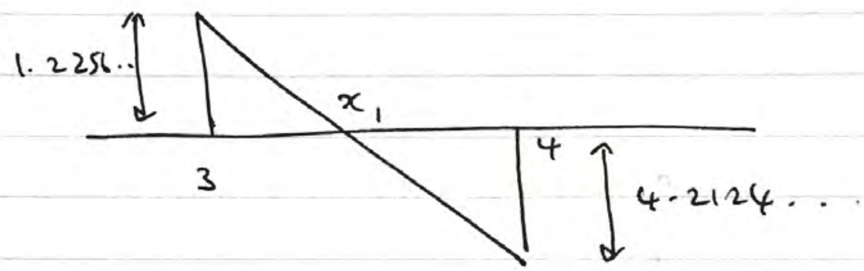
b) $f(x) = 6x^{\frac{1}{2}} - x^2 - \frac{1}{2}x^{-1}$

$f'(x) = 3x^{-\frac{1}{2}} - 2x + \frac{1}{2}x^{-2}$

$f(3) = 1.2256...$
$f'(3) = -4.2124...$

$\therefore x_1 = 3 - \dfrac{1.2256...}{-4.2124...} = 3.29096.$
$= \boxed{3.291}$ 3d.p.

c)



$\dfrac{4-x_1}{x_1-3} = \dfrac{4.2124..}{1.2256..}$ let $\dfrac{4.2124..}{1.2256..} = c$

then $\dfrac{4-x_1}{x_1-3} = c$

$\Rightarrow 4 - x_1 = cx_1 - 3c$

$\Rightarrow x_1(c+1) = 4 + 3c$

$\Rightarrow x_1 = \dfrac{4+3c}{c+1} = \boxed{3.229}$ (3dp).

Q2a) $x^2 - \frac{4}{5}x + \frac{2}{5} = 0$

$\therefore \alpha\beta = \frac{2}{5}$ and $\alpha + \beta = \frac{4}{5}$

b) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{4}{5}\right)^2 - 2\left(\frac{2}{5}\right) = \boxed{-\frac{4}{25}} = -0.16$

c) $\left(x - \frac{1}{\alpha^2}\right)\left(x - \frac{1}{\beta^2}\right) = 0$

$x^2 - x\left(\frac{1}{\beta^2} + \frac{1}{\alpha^2}\right) + \frac{1}{\alpha^2\beta^2} = 0$

$x^2 - x\left(\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right) + \frac{1}{\left(\frac{2}{5}\right)^2} = 0$

$x^2 - x\left(\frac{-\frac{4}{25}}{\left(\frac{2}{5}\right)^2}\right) + \frac{1}{\frac{4}{25}} = 0$

$x^2 + x + \frac{25}{4} = 0$

$\underline{\times 4}$: $\boxed{4x^2 + 4x + 25 = 0}$

Q3a) $A = \begin{pmatrix} 6 & 4 \\ 1 & 1 \end{pmatrix}$

$\det A = 6 - 4 = 2 \neq 0$

hence A is non-singular.

b) $\text{Area}_{\text{image}} = \text{Area}_{\text{object}} \times |\det M|$.

$\text{Area}_S = 10 \times |\det M| = 10 \times 2 = \boxed{20}$ units^2.

c) $BR = T$

$\text{Area}_T = \text{Area}_R \times |\det B| = 10 \times |\det B|$.

$$A^2 = \begin{pmatrix} 6 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 40 & 28 \\ 7 & 5 \end{pmatrix}$$

$\det |A^2| = 40(5) - 7(28) = 4$

hence $\text{Area}_T = 4^2 \times 10 = \boxed{160}$

$$\boxed{\text{Remember, } \det(A^4) = [\det(A)]^4}$$

in general, $\det(A^n) = [\det(A)]^n$

Q4a) $(x+4)$ and $(x-3)$ are factors.

hence $f(x) = (x+4)(x-3)(x^2+ax+b)$

$$(x+4)(x-3) = x^2+x-12$$

$$\begin{array}{r} x^2+2x+5 \\ x^2+x-12 \,\overline{)\, x^4+3x^3-5x^2-19x-60} \\ x^4+x^3-12x^2 \\ \hline 0+2x^3+7x^2-19x \\ 2x^3+2x^2-24x \\ \hline 0+5x^2+5x-60 \\ 5x^2+5x-60 \\ \hline 0 \quad 0 \quad 0 \end{array}$$

//

$\therefore\ f(x) = (x+4)(x+3)(x^2+2x+5) = 0$

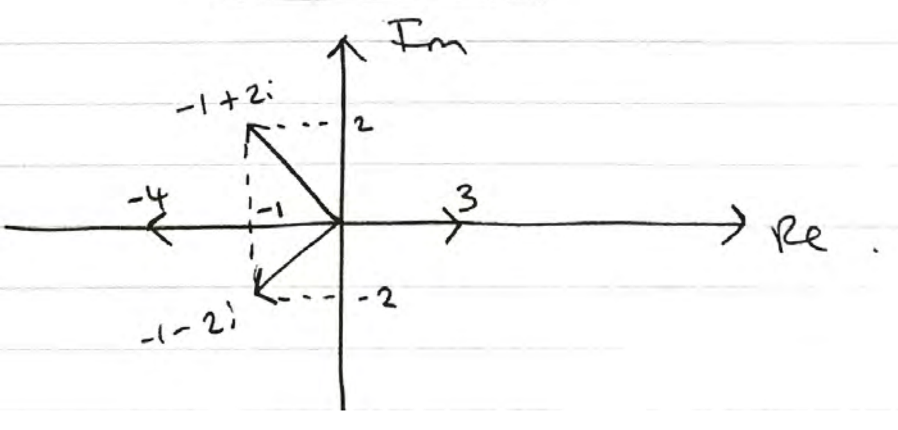
$$x^2+2x+5 = 0$$

$\left.\begin{array}{l} a=1 \\ b=2 \\ c=5 \end{array}\right\}$ $x = \dfrac{-2 \pm \sqrt{4-4(1)(5)}}{2} = \dfrac{-2 \pm 4i}{2}$

$= -1 \pm 2i$ //

$$\boxed{\begin{array}{l} x = -1+2i \\ x = -1-2i \end{array}}$$

b)



Q5a) $\sum_{1}^{n}(9r^2-4r) = 9\sum_{1}^{n}r^2 - 4\sum_{1}^{n}r$

$= \frac{9n}{6}(n+1)(2n+1) - \frac{4n}{2}(n+1)$

$= \frac{3n}{2}(n+1)(2n+1) - 2n(n+1)$

$= \frac{n}{2}(n+1)\left[3(2n+1) - 4\right] = \boxed{\frac{n}{2}(n+1)(6n-1)}$

b) $\sum_{1}^{12}(9r^2 - 4r + k(2^r)) = 6630$

$\sum_{1}^{12}(9r^2-4r) + k\sum_{1}^{12}2^r = 6630$

$\sum_{1}^{12}2^r$ — Geometric series; $a = 2$

$[2 + 2(2) + 2(2)^2 + .]$ $r = 2$

Sum $= \frac{2(1-(2)^{12})}{1-2} = 8190$ //

$\therefore$ and $\sum_{1}^{12}9r^2-4r = \frac{12}{2}(13)(6(12)-1) = 5538$ //

$\Rightarrow 5538 + 8190k = 6630$

$\Rightarrow k = \frac{6630-5538}{8190} = \boxed{\frac{2}{15}}$

Q6ia) B^{-1}: $\det B = -1(-4) - 3(2) = -2$

$$\therefore B^{-1} = -\frac{1}{2}\begin{pmatrix} -4 & -2 \\ -3 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}}$$

b) $Y = AB$

$YB^{-1} = ABB^{-1}$

$$YB^{-1} = A \;/\!/ = \begin{pmatrix} 4 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 8-3 & 4-1 \\ 2+0 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}}$$

iia) $$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$M = \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix} = \begin{pmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{pmatrix}$$

so: $k\cos\theta = -\sqrt{3}$ ~ 1

$k\sin\theta = 1$ — 2

$1^2 + 2^2$: $k^2\cos^2\theta + k^2\sin^2\theta = 3+1=4$

$\therefore (k^2)(\sin^2\theta + \cos^2\theta) = 4$

$k^2 = 4$

$\boxed{k = 2}$ $(k > 0)$

now $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right) = 30°, 150°$ $\theta = 150°, -150°$

the value of θ that satisfies **both** 1 and 2 is $150°$

hence $\boxed{\theta = 150°}$

Q7 i)

$$\frac{2w-3}{10} = \frac{(4+7i)(4+3i)}{(4-3i)(4+3i)}$$

$$\frac{2w-3}{10} = \frac{16+12i+28i-21}{16+9} = \frac{-5+40i}{25}$$

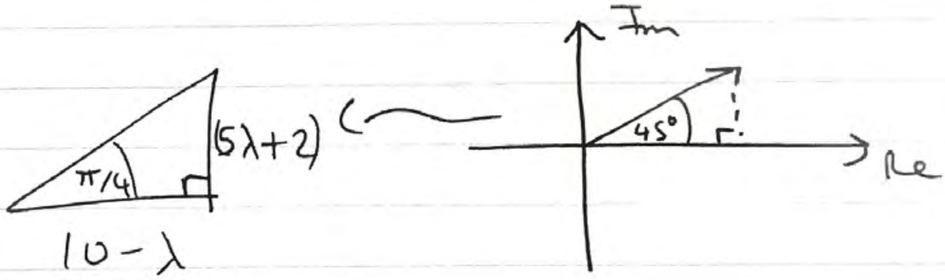
$$\therefore \frac{2w-3}{10} = -\frac{1}{5} + \frac{8}{5}i$$

$$\overset{\times 10}{\Rightarrow} \quad 2w-3 = -2+16i$$

$$\Rightarrow \quad 2w = 1+16i$$

$$\Rightarrow \quad \boxed{w = \frac{1}{2} + 8i}$$

ii) $z = 10 + 2i + 5\lambda i - \lambda = (10-\lambda) + (5\lambda+2)i$ //



$$\therefore \tan\frac{\pi}{4} = \frac{5\lambda+2}{10-\lambda} = 1$$

$$\Rightarrow \quad 5\lambda + 2 = 10 - \lambda$$

$$\Rightarrow \quad 6\lambda = 8$$

$$\Rightarrow \quad \boxed{\lambda = \frac{4}{3}}$$

Q8a) $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a \quad \therefore \quad \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2ap} = \frac{1}{p}$ //

$\therefore \quad y - 2ap = \frac{1}{p}(x - ap^2)$

$\Rightarrow y = \frac{1}{p}x - ap + 2ap$

$\xrightarrow{\times p} \quad py = x + ap^2$ $\square$

b) $x = -a,\ y = \frac{5}{6}a$: $p\left(\frac{5a}{6}\right) = -a + ap^2$

$\therefore \quad \frac{5p}{6} = -1 + p^2$

$\times 6$: $\quad 5p + 6 = 6p^2$

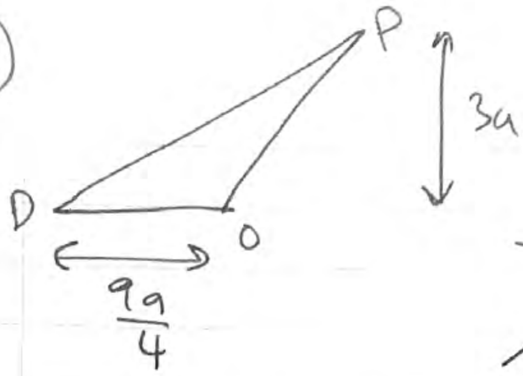
$\Rightarrow \quad 6p^2 - 5p - 6 = 0$

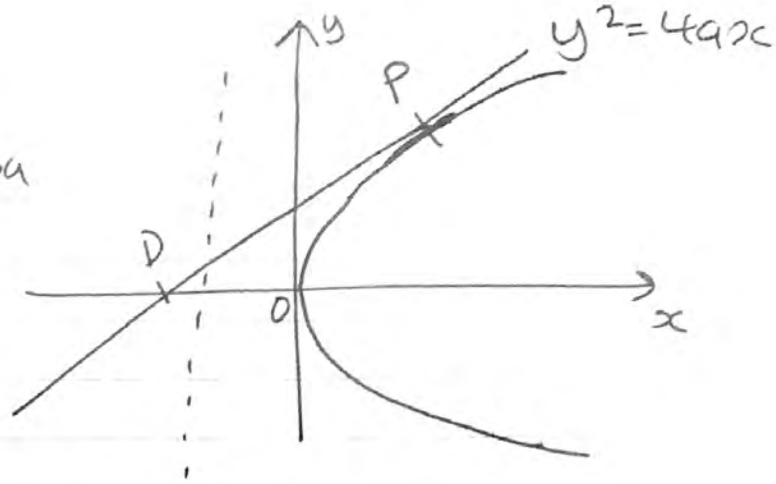
By Quadratic formula ; $\quad p = \frac{3}{2}$ // , $\underbrace{p = -\frac{2}{3}}_{\text{reject } (p>0)}$

at x-axis, $y = 0$: $x + ap^2 = 0$

$x = -a\left(\frac{3}{2}\right)^2 = \boxed{-\frac{9a}{4}}$

c)





y-coordinate of P: $2a\left(\frac{3}{2}\right) = 3a$ //

$\therefore$ Area $= \frac{1}{2}\left(\frac{9a}{4}\right)(3a) = \boxed{\frac{27a^2}{8}}$

Q9) $\underline{n=1}$: $f(1) = 7 - 2 = 5 = (5) \times 1$ //

$\therefore$ true for $n = 1$.

assume true for $n = k$, ie $f(k)$ is divisible by 5.

Consider $n = k+1$:

$f(k+1) = 7^{k+1} - 2^{k+1}$

$= 7(7^k) - 2(2^k)$

$= 7\left[7^k - 2^k\right] - 5(2^k)$

$= 7f(k) - 5(2^k)$

$\therefore$ true for $n = k+1$.

So, true for $n=1$.
true for $n=k+1$ when assumed true for $n=k$.
$\therefore$ By Mathematical Induction true for all $n \in \mathbb{Z}^+$